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Weil-Petersson volume of moduli spaces, Mirzakhani's recursion and matrix models

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Abstract

We prove that Mirzakhani's recursions for the volumes of moduli space of Riemann surfaces are a special case of random matrix recursion relations, and therefore we confirm again that Kontsevich's integral is a generating function for those volumes. As an application, we propose a formula for the Weil-Petersson volume $\mathrm{Vol}(\mathcal{M}_{g,0})$.

1 Introduction

Let

$$\begin{aligned} V_{g,n}(L_1,\dots,L_n) &= \mathrm{Vol}(\mathcal{M}_{g,n}) \\ &= \sum_{d_0+\dots+d_n=3g-3+n} \left(\prod_{i=0}^{n} \frac{1}{d_i!}\right) \left\langle \kappa_1^{d_0}\tau_{d_1}\dots\tau_{d_n}\right\rangle_{g,n} L_1^{2d_1}\dots L_n^{2d_n} \end{aligned} \quad (1-1)$$

denote the volume of the moduli space of curves of genus g, with n geodesic boundaries of lengths $L_1,\dots,L_n$, measured with the Weil-Petersson metrics. Using Teichmuller pants decomposition and hyperbolic geometry, M. Mirzakhani [4] has found a recursion relation among the $V_{g,n}$'s, which allows to compute all of them in a recursive manner. It was then observed [5] that this recursion relation is equivalent to Virasoro constraints.

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In fact, Mirzakhani's recursion relation takes a form [3] which is amazingly similar to the recursion relation obeyed by matrix models correlation functions ([1, 2]) and which were indeed initialy derived from loop equations [1], i.e. Virasoro constraints.

Here we make this observation more precise, and we prove that after Laplace transform, Mirzakhani's recursion is **identical** to the recursion of [2] for the Kontsevich integral with times (Kontsevich's integral depends only on odd times):

$$Z(t_k) = \int dM\, \mathrm{e}^{-N\,\mathrm{Tr}\,[\frac{M^3}{3}+\Lambda M^2]} \qquad , \qquad t_{2k+3} = \frac{1}{N}\,\mathrm{Tr}\,\Lambda^{-(2k+3)} = \frac{(2\pi)^{2k}\,(-1)^k}{(2k+1)!} + 2\delta_{k,0}. \tag{1-2}$$

2 Laplace transform

Define the Laplace transforms of the $V_{g,n}$'s:

$$\begin{aligned} & W_n^g(z_1,\dots,z_n) \\ = \quad & 2^{-m_{g,n}} \int_0^\infty dL_1\dots dL_n \mathrm{e}^{-\sum_i z_i L_i} \prod_{i=1}^n L_i\;\; V_{g,n}(L_1,\dots,L_n) \\ = \quad & 2^{-m_{g,n}} \sum_{d_0+\dots+d_n=3g-3+n} \left(\prod_{i=0}^n \frac{1}{d_i!}\right) \left\langle \kappa_1^{d_0} \tau_{d_1}\dots\tau_{d_n}\right\rangle_{g,n} \frac{(2d_1+1)!}{z_1^{2d_1+2}} \cdots \frac{(2d_n+1)!}{z_n^{2d_n+2}} \end{aligned} \tag{2-1}$$

where (see [4]) $m_{g,n}=\delta_{g,1}\delta_{n,1}$.

Since the $V_{g,n}$'s are even polynomials of the L_i's, of degree $2d_{g,n}$ where

$$d_{g,n} = \dim \mathcal{M}_{g,n} = 3g-3+n \tag{2-2}$$

the W_n^g's are even polynomials of the $1/z_i$'s of degree $2d_{g,n}+2$. Let us also define:

$$W_1^0 = 0 \tag{2-3}$$

$$W_2^0(z_1,z_2) = \frac{1}{(z_1-z_2)^2} \tag{2-4}$$

and

$$dE_u(z) = \frac{1}{2}\left(\frac{1}{z-u} - \frac{1}{z+u}\right). \tag{2-5}$$

We prove the following theorems:

Theorem 2.1 *For any $2g-2+n+1>0$, the W_{n+1}^g satisfy the recursion relation*

$$\boxed{\begin{aligned} W_{n+1}^g(z,K) \quad = \quad \mathrm{Res}_{u\to 0} \tfrac{\pi dE_u(z)}{u\sin 2\pi u} \Big[& \textstyle\sum_{h=0}^g \sum_{J\subset K} W_{1+|J|}^h(u,J) W_{1+n-|J|}^{g-h}(-u,K/J) \\ & + W_{n+2}^{g-1}(u,-u,K) \Big] \end{aligned}} \tag{2-6}$$

where the RHS includes all possible W_k^h, including $W_1^0 = 0$ and W_2^0, and where

$$K = \{z_1, \dots, z_n\} \tag{2-7}$$

is a set of n variables.

proof:

This relation is merely the Laplace transform of Mirzakhani's recursion. See the appendix for a detailed proof. □

Corollary 2.1 *W_n^g are the invariants defined in [2] for the curve:*

$$\left\{ \begin{array}{l} x(z) = z^2 \\ -2y(z) = \frac{\sin(2\pi z)}{2\pi} = z - 2\frac{\pi^2}{3} z^3 + \frac{2\pi^4}{15} z^5 - \frac{4\pi^6}{315} z^7 + \frac{2\pi^8}{2835} z^9 + \dots \end{array} \right. \tag{2-8}$$

which is a special case of Kontsevich's curve:

$$Z(t_k) = \int dM \, \mathrm{e}^{-N \operatorname{Tr}\left[\frac{M^3}{3} + \Lambda M^2\right]} \qquad , \qquad t_k = \frac{1}{N} \operatorname{Tr} \Lambda^{-k} = \frac{(2\pi)^{k-3} \, \sin(\pi k/2)}{(k-2)!} \tag{2-9}$$

For instance we have:

$$\ln Z(t_k) = \sum_{g=0}^{\infty} N^{2-2g} W_0^g \tag{2-10}$$

(W_0^g is often noted $-F_g$ in the litterature).

proof:

Eq. 2-6 is precisely the definiton of the invariants of [2] for the curve

$$\left\{ \begin{array}{l} x(z) = z^2 \\ -2y(z) = \frac{\sin(2\pi z)}{2\pi} = z - 2\frac{\pi^2}{3} z^3 + \frac{2\pi^4}{15} z^5 - \frac{4\pi^6}{315} z^7 + \frac{2\pi^8}{2835} z^9 + \dots \end{array} \right. \tag{2-11}$$

And it was proved in [2] that this curve is a special case of Kontsevich's curve:

$$\left\{ \begin{array}{l} x(z) = z^2 \\ y(z) = z - \frac{1}{2} \sum_{j=0}^{\infty} t_{j+2} z^j \end{array} \right. \tag{2-12}$$

which corresponds to the computation of the topological expansion of the Kontsevich integral:

$$Z(t_k) = \int dM \, \mathrm{e}^{-N \operatorname{Tr}\left[\frac{M^3}{3} + \Lambda M^2\right]} \qquad , \qquad t_k = \frac{1}{N} \operatorname{Tr} \Lambda^{-k} \tag{2-13}$$

$$\ln Z(t_k) = -\sum_{g=0}^{\infty} N^{2-2g} F_g \tag{2-14}$$

□

Theorem 2.2 *For any $2g-2+n>0$ we have:*

$$(2g-2+n)\, W_n^g(K) = \frac{1}{4\pi^2}\, \operatorname*{Res}_{u\to 0} \left(u\cos(2\pi u) - \frac{1}{2\pi}\sin(2\pi u)\right)\, W_{n+1}^g(u,K) \tag{2-15}$$

or in inverse Laplace transform:

$$\boxed{(2g-2+n)\, V_{g,n}(K) = \frac{1}{2i\pi}\, V'_{g,n+1}(K,2i\pi)} \tag{2-16}$$

where $'$ means the derivative with respect to the $n+1^{\text{th}}$ variable.

proof:

This is a mere application of theorem 4.7. in [2], as well as its Laplace transform. □

In particular with $n=0$ we get:

$$V_{g,0} = \mathrm{Vol}(\mathcal{M}_{g,0}) = \frac{1}{2g-2}\, \frac{V'_{g,1}(2i\pi)}{2i\pi} \tag{2-17}$$

for instance for $g=2$:

$$V_{2,0} = \frac{43\pi^6}{2160}. \tag{2-18}$$

2.1 Examples

From [4] we get:

$$W_3^0 = \frac{1}{z_1^2 z_2^2 z_3^2} \tag{2-19}$$

$$W_1^1 = \frac{1}{8z_1^4} + \frac{\pi^2}{12z_1^2} \tag{2-20}$$

$$W_4^0 = \frac{1}{z_1^2 z_2^2 z_3^2 z_4^2}\left(2\pi^2 + 3(\frac{1}{z_1^2}+\frac{1}{z_2^2}+\frac{1}{z_3^2}+\frac{1}{z_4^2})\right) \tag{2-21}$$

$$W_2^1 = \frac{1}{z_1^2 z_2^2}\left(\frac{\pi^4}{4} + \frac{\pi^2}{2}(\frac{1}{z_1^2}+\frac{1}{z_2^2}) + \frac{5}{8z_1^4} + \frac{5}{8z_2^4} + \frac{3}{8z_1^2 z_2^2}\right) \tag{2-22}$$

$$W_5^0 = \frac{1}{z_1^2 z_2^2 z_3^2 z_4^2 z_5^2}\left(10\pi^4 + 18\pi^2\sum_i \frac{1}{z_i^2} + 15\sum_i \frac{1}{z_i^4} + 18\sum_{i<j}\frac{1}{z_i^2 z_j^2}\right) \tag{2-23}$$

$$W_1^2 = \frac{1}{192z_1^2}\left(29\pi^8 + \frac{338\pi^6}{5z_1^2} + \frac{139\pi^4}{z_1^4} + \frac{203\pi^2}{z_1^6} + \frac{315}{2z_1^8}\right) \tag{2-24}$$

Those functions are the same as those which appear in section 10.4.1 of [2], for the Kontsevich curve with times:

$$t_3 - 2 = 1, t_5 = -\frac{2\pi^2}{3}, t_7 = \frac{2\pi^4}{15}, t_9 = -\frac{4\pi^6}{315}, t_{11} = \frac{2\pi^8}{2835}, \dots \tag{2-25}$$

i.e. the rational curve:

$$\mathcal{E}_K = \begin{cases} x(z) = z^2 \\ -2y(z) = \frac{\sin(2\pi z)}{2\pi} = z - 2\frac{\pi^2}{3}z^3 + \frac{2\pi^4}{15}z^5 - \frac{4\pi^6}{315}z^7 + \frac{2\pi^8}{2835}z^9 + \dots \end{cases} \tag{2-26}$$

It is to be noted that those t_k's are closely related to the β_k's of [5, 3].

3 Conclusion

We have shown that, after Laplace transform, Mirzakhani's recursions are nothing but the solution of loop equations (i.e. Virasoro constraints) for the Kontsevich integral with some given set of times. It would be interesting to understand what the invariants of [2] compute for an arbitrary spectral curve (for instance for other Kontsevich times).

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Appendix A Laplace transform of the equations

Let us write:

$$L_K = \{L_1, \dots, L_n\} \tag{1-1}$$

$$H_n^g(x, y, L_K) = xyV_{g-1,n+2}(x, y, L_K) + \sum_{h=0}^{g} \sum_{J\in K} xV_{h,1+|J|}(x, L_J)yV_{g-h,n+1-|J|}(y, L_{K/J}) \tag{1-2}$$

where all the $V_{h,k}$ terms in the RHS are such that $2h + k - 2 > 0$ (i.e. stable curves only), as well as their laplace transform:

$$\widetilde{H}_n^{(g)}(z, z', L_K) := \int_0^\infty dx \int_0^\infty dye^{-zx}e^{-z'y}H_n^g(x, y, L_K). \tag{1-3}$$

Mirzakhani's recursion reads:

$$\begin{aligned}
2LV_{g,n+1}(L,L_K) &= \int_0^L dt \int_0^\infty dx \int_0^\infty dy K(x+y,t) H_n^g(x,y,L_K) \\
&+ \sum_{m=1}^{n} \int_0^L dt \int_0^\infty dx (K(x,t+L_m)+K(x,t-L_m)) x V_{g,n-1}(x,\hat{L}_m)
\end{aligned} \tag{1-4}$$

where

$$K(x,t) = \frac{1}{1+\mathrm{e}^{\left(\frac{x+t}{2}\right)}} + \frac{1}{1+\mathrm{e}^{\left(\frac{x-t}{2}\right)}} \tag{1-5}$$

and $\hat{L}_m = L_K/\{L_m\}$.

Let $\tilde{H}_n^g$ be the Laplace transform of H_n^g with respect to x and y.

The Laplace transform of the first term in eq.1-4 is:

$$\begin{aligned}
& \sum_{\epsilon=\pm 1} \int_0^\infty dL\, \mathrm{e}^{-zL} \int_0^L dt \int_0^\infty dx \int_0^\infty dy \frac{1}{1+\mathrm{e}^{\frac{x+y+\epsilon t}{2}}}\ H_n^g(x,y,L_K) \\
= \quad & \sum_{\epsilon=\pm 1} \int_0^\infty dt \int_t^\infty dL\, \mathrm{e}^{-zL} \int_0^\infty dx \int_0^\infty dy \frac{1}{1+\mathrm{e}^{\frac{x+y+\epsilon t}{2}}}\ H_n^g(x,y,L_K) \\
= \quad & \sum_{\epsilon=\pm 1} \frac{1}{z} \int_0^\infty dt\, \mathrm{e}^{-zt} \int_0^\infty dx \int_0^\infty dy \frac{1}{1+\mathrm{e}^{\frac{x+y+\epsilon t}{2}}}\ H_n^g(x,y,L_K) \\
= \quad & -\sum_{j=1}^\infty \frac{1}{z} \int_0^\infty dt\, \mathrm{e}^{-zt} \int_0^\infty dx \int_0^\infty dy (-1)^j \mathrm{e}^{-\frac{j}{2}(x+y+t)}\ H_n^g(x,y,L_K) \\
& +\sum_{j=0}^\infty \frac{1}{z} \int_0^\infty dx \int_0^\infty dy \int_{x+y}^\infty dt\, \mathrm{e}^{-zt} (-1)^j \mathrm{e}^{\frac{j}{2}(x+y-t)}\ H_n^g(x,y,L_K) \\
& -\sum_{j=1}^\infty \frac{1}{z} \int_0^\infty dx \int_0^\infty dy \int_0^{x+y} dt\, \mathrm{e}^{-zt} (-1)^j \mathrm{e}^{-\frac{j}{2}(x+y-t)}\ H_n^g(x,y,L_K) \\
= \quad & -\sum_{j=1}^\infty \frac{1}{z} \int_0^\infty dx \int_0^\infty dy \frac{(-1)^j}{z+\frac{j}{2}} \mathrm{e}^{-\frac{j}{2}(x+y)}\ H_n^g(x,y,L_K) \\
& +\sum_{j=0}^\infty \frac{1}{z} \int_0^\infty dx \int_0^\infty dy \frac{(-1)^j}{z+\frac{j}{2}} \mathrm{e}^{-z(x+y)}\ H_n^g(x,y,L_K) \\
& -\sum_{j=1}^\infty \frac{1}{z} \int_0^\infty dx \int_0^\infty dy \frac{(-1)^j}{z-\frac{j}{2}} (1-\mathrm{e}^{-(z-\frac{j}{2})(x+y)}) \mathrm{e}^{-\frac{j}{2}(x+y)}\ H_n^g(x,y,L_K) \\
= \quad & -2\sum_{j=1}^\infty \frac{(-1)^j}{z^2-\left(\frac{j}{2}\right)^2}\ \tilde{H}_n^g(\frac{j}{2},\frac{j}{2},L_K) + \frac{1}{z^2}\ \tilde{H}_n^g(z,z,L_K) \\
& +2\sum_{j=1}^\infty \frac{(-1)^j}{z^2-\left(\frac{j}{2}\right)^2}\ \tilde{H}_n^g(z,z,L_K) \\
= \quad & -2\sum_{j=1}^\infty \frac{(-1)^j}{z^2-\left(\frac{j}{2}\right)^2}\ \tilde{H}_n^g(\frac{j}{2},\frac{j}{2},L_K) + \frac{2\pi}{z\sin 2\pi z}\ \tilde{H}_n^g(z,z,L_K)
\end{aligned}$$

$$
\begin{aligned}
&= \left(\operatorname*{Res}_{u\to z} + \sum_{j=1}^{\infty}\operatorname*{Res}_{u\to\pm\frac{j}{2}}\right)\frac{du}{u-z}\,\frac{2\pi}{u\sin(2\pi u)}\;\tilde{H}^g_n(u,u,L_K)\\
&= \operatorname*{Res}_{u\to 0}\frac{du}{z-u}\,\frac{2\pi}{u\sin(2\pi u)}\;\tilde{H}^g_n(u,u,L_K)\\
&= \operatorname*{Res}_{u\to 0}\frac{2\pi\,du}{u\sin(2\pi u)}\,dE_u(z)\;\tilde{H}^g_n(u,u,L_K)
\end{aligned}
$$

$(1-6)$

Using the notation

$$R(x,t,L_m) := (K(x,t+L_m)+K(x,t-L_m)), \tag{1-7}$$

the Laplace transform of the second term in eq.1-4 is:

$$
\begin{aligned}
& \int_0^\infty dL_m \mathrm{e}^{-z_m L_m}\int_0^\infty dL\,\mathrm{e}^{-zL}\int_0^L dt\int_0^\infty dx R(x,t,L_m) xV_{g,n-1}(x,\hat{L}_m)\\
=\;& \frac{1}{z}\int_0^\infty dx\int_0^\infty dL_m\mathrm{e}^{-z_mL_m}\int_0^\infty dt\,\mathrm{e}^{-zt}\,R(x,t,L_m)xV_{g,n-1}(x,\hat{L}_m)\\
=\;& \frac{1}{z}\int_0^\infty dx\int_0^\infty dL_m\mathrm{e}^{-z_mL_m}\int_{L_m}^\infty dt\,\mathrm{e}^{-z(t-L_m)}\,K(x,t)xV_{g,n-1}(x,\hat{L}_m)\\
&+\frac{1}{z}\int_0^\infty dx\int_0^\infty dL_m\mathrm{e}^{-z_mL_m}\int_{-L_m}^\infty dt\,\mathrm{e}^{-z(t+L_m)}\,K(x,t)xV_{g,n-1}(x,\hat{L}_m)\\
=\;& \frac{1}{z}\int_0^\infty dx\int_0^\infty dt\,\mathrm{e}^{-zt}\int_0^t dL_m\mathrm{e}^{-(z_m-z)L_m}\,K(x,t)xV_{g,n-1}(x,\hat{L}_m)\\
&+\frac{1}{z}\int_0^\infty dx\int_0^\infty dt\,\mathrm{e}^{-zt}\int_0^\infty dL_m\mathrm{e}^{-(z_m+z)L_m}\,K(x,t)xV_{g,n-1}(x,\hat{L}_m)\\
&+\frac{1}{z}\int_0^\infty dx\int_{-\infty}^0 dt\,\mathrm{e}^{-zt}\int_{-t}^\infty dL_m\mathrm{e}^{-(z_m+z)L_m}\,K(x,t)xV_{g,n-1}(x,\hat{L}_m)\\
=\;& \frac{1}{z}\int_0^\infty dx\int_0^\infty dt\,\frac{\mathrm{e}^{-zt}-\mathrm{e}^{-z_mt}}{z_m-z}\,K(x,t)xV_{g,n-1}(x,\hat{L}_m)\\
&+\frac{1}{z}\int_0^\infty dx\int_0^\infty dt\,\frac{\mathrm{e}^{-zt}+\mathrm{e}^{z_mt}}{z_m+z}\,K(x,t)xV_{g,n-1}(x,\hat{L}_m)\\
&+\frac{1}{z}\int_0^\infty dx\int_0^\infty dt\,\frac{\mathrm{e}^{-z_mt}}{z_m+z}\,K(x,t)xV_{g,n-1}(x,\hat{L}_m)\\
=\;& \frac{1}{z}\int_0^\infty dx\int_0^\infty dt\left(\frac{\mathrm{e}^{-zt}-\mathrm{e}^{-z_mt}}{z_m-z}+\frac{\mathrm{e}^{-zt}+\mathrm{e}^{-z_mt}}{z_m+z}\right)K(x,t)xV_{g,n-1}(x,\hat{L}_m)\\
=\;& \frac{1}{z}\int_0^\infty dx\int_0^\infty dt\,\frac{2z_m\mathrm{e}^{-zt}-2z\mathrm{e}^{-z_mt}}{(z_m^2-z^2)}\,\frac{1}{1+\mathrm{e}^{\frac{x+t}{2}}}\,xV_{g,n-1}(x,\hat{L}_m)\\
&+\frac{1}{z}\int_0^\infty dx\int_0^x dt\,\frac{2z_m\mathrm{e}^{-zt}-2z\mathrm{e}^{-z_mt}}{(z_m^2-z^2)}\,\frac{1}{1+\mathrm{e}^{\frac{x-t}{2}}}\,xV_{g,n-1}(x,\hat{L}_m)\\
&+\frac{1}{z}\int_0^\infty dx\int_x^\infty dt\,\frac{2z_m\mathrm{e}^{-zt}-2z\mathrm{e}^{-z_mt}}{(z_m^2-z^2)}\,\frac{1}{1+\mathrm{e}^{\frac{x-t}{2}}}\,xV_{g,n-1}(x,\hat{L}_m)\\
=\;& -\sum_{j=1}^\infty\frac{(-1)^j}{z}\int_0^\infty dx\int_0^\infty dt\,\frac{2z_m\mathrm{e}^{-zt}-2z\mathrm{e}^{-z_mt}}{(z_m^2-z^2)}\,\mathrm{e}^{-\frac{j}{2}(x+t)}\,xV_{g,n-1}(x,\hat{L}_m)
\end{aligned}
$$

$$
\begin{aligned}
& -\sum_{j=1}^{\infty}\frac{(-1)^j}{z}\int_0^\infty dx\int_0^x dt\,\frac{2z_m\mathrm{e}^{-zt}-2z\mathrm{e}^{-z_m t}}{(z_m^2-z^2)}\,\mathrm{e}^{-\frac{j}{2}(x-t)}\,xV_{g,n-1}(x,\hat{L}_m)\\
& +\sum_{j=0}^{\infty}\frac{(-1)^j}{z}\int_0^\infty dx\int_x^\infty dt\,\frac{2z_m\mathrm{e}^{-zt}-2z\mathrm{e}^{-z_m t}}{(z_m^2-z^2)}\,\mathrm{e}^{\frac{j}{2}(x-t)}\,xV_{g,n-1}(x,\hat{L}_m)\\
= & -\sum_{j=1}^{\infty}\frac{(-1)^j}{z}\int_0^\infty dx\,\frac{\frac{2z_m}{z+\frac{j}{2}}-\frac{2z}{z_m+\frac{j}{2}}}{(z_m^2-z^2)}\,\mathrm{e}^{-\frac{j}{2}x}\,xV_{g,n-1}(x,\hat{L}_m)\\
& -\sum_{j=1}^{\infty}\frac{(-1)^j}{z}\int_0^\infty dx\int_0^x dt\,\frac{2z_m\,\frac{\mathrm{e}^{-\frac{j}{2}x}-\mathrm{e}^{-zx}}{z-\frac{j}{2}}-2z\,\frac{\mathrm{e}^{-\frac{j}{2}x}-\mathrm{e}^{-z_m x}}{z_m-\frac{j}{2}}}{(z_m^2-z^2)}\,xV_{g,n-1}(x,\hat{L}_m)\\
& +\sum_{j=0}^{\infty}\frac{(-1)^j}{z}\int_0^\infty dx\int_x^\infty dt\,\frac{\frac{2z_m\mathrm{e}^{-zx}}{z+\frac{j}{2}}-\frac{2z\mathrm{e}^{-z_m x}}{z_m+\frac{j}{2}}}{(z_m^2-z^2)}\,xV_{g,n-1}(x,\hat{L}_m)\\
= & -2\sum_{j=1}^{\infty}\frac{(-1)^j}{z}\,\frac{z+z_m+\frac{j}{2}}{(z_m+z)(z+\frac{j}{2})(z_m+\frac{j}{2})}\,W_{g,n-1}(\frac{j}{2},\hat{L}_m)\\
& -2\sum_{j=1}^{\infty}\frac{(-1)^j}{z}\,\frac{z+z_m-\frac{j}{2}}{(z_m+z)(z-\frac{j}{2})(z_m-\frac{j}{2})}\,W_{g,n-1}(\frac{j}{2},\hat{L}_m)\\
& +2\sum_{j=1}^{\infty}\frac{(-1)^j}{z}\,\frac{z_m}{(z-\frac{j}{2})(z_m^2-z^2)}\,W_{g,n-1}(z,\hat{L}_m)\\
& -2\sum_{j=1}^{\infty}(-1)^j\,\frac{1}{(z_m-\frac{j}{2})(z_m^2-z^2)}\,W_{g,n-1}(z_m,\hat{L}_m)\\
& +2\sum_{j=0}^{\infty}\frac{(-1)^j}{z}\,\frac{z_m}{(z+\frac{j}{2})(z_m^2-z^2)}\,W_{g,n-1}(z,\hat{L}_m)\\
& -2\sum_{j=0}^{\infty}(-1)^j\,\frac{1}{(z_m+\frac{j}{2})(z_m^2-z^2)}\,W_{g,n-1}(z_m,\hat{L}_m)\\
= & -4\sum_{j=1}^{\infty}\operatorname*{Res}_{u\to\pm j}\frac{\pi du}{\sin 2\pi u}\,\frac{1}{z}\,\frac{z+z_m+u}{(z_m+z)(z+u)(z_m+u)}\,W_{g,n-1}(u,\hat{L}_m)\\
& +4\,\frac{z_m\pi}{z\sin(2\pi z)\,(z_m^2-z^2)}\,W_{g,n-1}(z,\hat{L}_m)\\
& -4\,\frac{\pi}{\sin(2\pi z_m)\,(z_m^2-z^2)}\,W_{g,n-1}(z_m,\hat{L}_m)\\
= & -4\sum_{j=1}^{\infty}\operatorname*{Res}_{u\to\pm\frac{j}{2}}\frac{\pi du}{\sin 2\pi u}\,\frac{z_m}{(z^2-u^2)(z_m^2-u^2)}\,W_{g,n-1}(u,\hat{L}_m)\\
& -4\operatorname*{Res}_{u\to z,z_m}\frac{\pi du}{\sin 2\pi u}\,\frac{z_m}{(z_m^2-u^2)(z^2-u^2)}\,W_{g,n-1}(u,\hat{L}_m)\\
= & \;4\operatorname*{Res}_{u\to 0}\frac{\pi du}{\sin 2\pi u}\,\frac{z_m}{(z^2-u^2)(z_m^2-u^2)}\,W_{g,n-1}(u,\hat{L}_m)\\
= & \;2\operatorname*{Res}_{u\to 0}\frac{\pi du}{2u\sin 2\pi u}\left(\frac{1}{z-u}-\frac{1}{z+u}\right)\left(\frac{1}{z_m-u}+\frac{1}{z_m+u}\right)W_{g,n-1}(u,\hat{L}_m)
\end{aligned}
$$

$$= \quad 4 \operatorname*{Res}_{u\to 0} \frac{\pi du}{2u\sin 2\pi u} \left(\frac{1}{z-u} - \frac{1}{z+u}\right) \frac{1}{z_m - u} W_{g,n-1}(u, \hat{L}_m) \qquad (1-8)$$

After taking the derivative with respect to z_m that gives the expected term:

$$\operatorname*{Res}_{u\to 0} \frac{2\pi du}{u\sin 2\pi u}\, dE_u(z)\; 2\, W_2^0(u, z_m) W_{g,n-1}(u, \hat{L}_m) \tag{1-9}$$

and therefore the Laplace transform of Eq. (1-4) gives the relation Eq. (2-6).

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